

A Necessary and Sufficient Condition for a Product Relation to Be Total

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Communicated by Gian-Carlo Rota

Received September 24, 1982

Let A, B , and C be sets, let ρ be a relation on $A \times B$, and let σ be a relation on $B \times C$. A necessary and sufficient condition for $\rho \circ \sigma$ to be total is provided in terms of a DeMorgan algebra defined on B . © 1984 Academic Press, Inc.

1. INTRODUCTION

Let A, B , and C be sets, let ρ be a relation on $A \times B$, and let σ be a relation on $B \times C$. The product (composite) relation $\rho \circ \sigma$ on $A \times C$ is the relation

$$\{ \langle a, c \rangle \mid \exists b \in B : a\rho b \text{ and } b\sigma c \}.$$

We are concerned here with determining when $\rho \circ \sigma$ is total, that is, when $\rho \circ \sigma = A \times C$. The product of the relations in Fig. 1a is not total, but the product in Fig. 1b is. Theorem 2 below provides a necessary and sufficient condition for totality in terms of a DeMorgan algebra defined on B . The result plays a fundamental role in a combinatorial theory [7] that seeks to extend classical logic to handle dynamic situations.

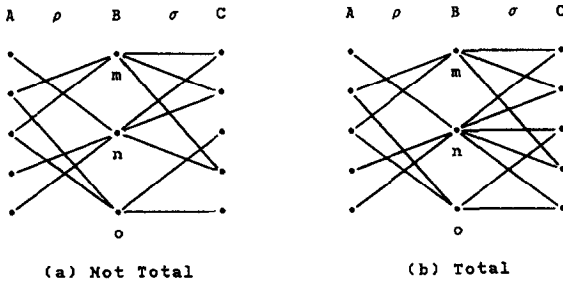


FIG. 1. Product relations.

We adopt the following notation. $\mathcal{P}(A)$ denotes the power set (set of all subsets) of the set A . For a relation ρ on $A \times B$ and for $a \in A$ and $Q \subseteq A$,

$$\begin{aligned} \rho(a) &= \{b \in B \mid a\rho b\} \\ \rho(Q) &= \{b \in B \mid \exists a \in Q: a\rho b\} \end{aligned}$$

ρ^{-1} denotes the converse of ρ . $\min(\mathcal{O})$ denotes the family of minimal sets, with respect to set inclusion, of the family of sets \mathcal{O} .

2. DEMORGAN ALGEBRAS

Two algebras are defined below that satisfy most of, but not all, the usual properties of a Boolean algebra. Algebras satisfying this reduced set of properties are called DeMorgan algebras [1-3, 5, 6, 8-11].

DEFINITION. A *DeMorgan algebra* is an algebraic system $\langle D, \vee, \wedge, \bar{} \rangle$ where \vee and \wedge are binary operations and $\bar{}$ is a unary operation such that:

1. $\langle D, \vee, \wedge \rangle$ forms a distributive lattice,
2. $\overline{(x \vee y)} = \bar{x} \wedge \bar{y}$ and $\overline{(x \wedge y)} = \bar{x} \vee \bar{y}$ (DeMorgan's Laws), and
3. $\bar{\bar{x}} = x$.

PROPERTY 1. For elements x and y in a DeMorgan algebra, $x \leq y \Leftrightarrow \bar{y} \leq \bar{x}$.

A unary operation \sim on a set S such that $\sim \sim s = s$ for all $s \in S$ is called an involution of S . The unary operation of a DeMorgan algebra is thus an involution. The following result provides a method for constructing a DeMorgan algebra from an involution defined on an arbitrary set. (See Balbes and Dwinger [1, p. 212].)

THEOREM 1. Let S be a set with an involution \sim , and let $\bar{}$ be the unary operation on $\mathcal{P}(S)$ such that for all $X \subseteq S$,

$$\bar{X} = S - \{\sim s \mid s \in X\}.$$

Then $\langle \mathcal{P}(S), \cup, \cap, \bar{} \rangle$ is a DeMorgan algebra.

3. THE Ψ AND Ω SETS

In Theorem 1, let $S = \mathcal{P}(N)$ for an arbitrary set N and let \sim denote set-theoretic complement. Then by Theorem 1, $\langle \mathcal{P}(\mathcal{P}(N)), \cup, \cap, \bar{} \rangle$ is a

a DeMorgan algebra. We are interested here in a special subalgebra of this DeMorgan algebra.

DEFINITION. Let N be a set. $\Psi(N)$ is then the set of $X \subseteq \mathcal{P}(N)$ such that for all $P, Q \subseteq N: (P \in X \text{ and } P \subseteq Q) \Rightarrow Q \in X$.

In the terminology of Birkhoff [4, pp. 55–61], $\langle \Psi(N), \subseteq \rangle$ is the ring of all J -closed subsets (dual semi-ideals) of $\mathcal{P}(N)$. $\langle \Psi(N), \subseteq \rangle$ is isomorphic to the free distributive lattice generated by $|N|$ symbols, with O and I adjoined. To make $\langle \Psi(N), \cup, \cap \rangle$ into a DeMorgan algebra, a unary operation is added that is equivalent—on $\Psi(N)$ —to the operation defined in Theorem 1 when S is interpreted as $\mathcal{P}(N)$ and \sim as set-theoretic complement.

PROPERTY 2. Let N be a set and let $\bar{}$ be the unary operation on $\Psi(N)$ such that for all $X \in \Psi(N)$,

$$\bar{X} = \{Q \subseteq N \mid \forall P \in X: P \cap Q \neq \phi\}.$$

Then $\langle \Psi(N), \cup, \cap, \bar{} \rangle$ is a DeMorgan algebra.

EXAMPLE. The distributive lattice for $\Psi(\{m, n\})$ is shown in Fig. 2. Moreover,

$$\begin{aligned} \bar{\phi} &= \{\phi, \{m\}, \{n\}, \{m, n\}\} \\ \overline{\{\{m\}, \{m, n\}\}} &= \{\{m\}, \{m, n\}\} \\ \overline{\{\{n\}, \{m, n\}\}} &= \{\{n\}, \{m, n\}\} \\ \overline{\{\{m, n\}\}} &= \{\{m\}, \{n\}, \{m, n\}\}. \end{aligned}$$

In this finite example, it is clear that each element of $\Psi(N)$ is determined by its minimal sets. $\{\{m\}, \{m, n\}\}$, for example, is determined by $\{\{m\}\}$. This property is exploited to produce an algebra that is isomorphic to $\langle \Psi(N), \cup, \cap, \bar{} \rangle$ but whose elements have a more compact representation.

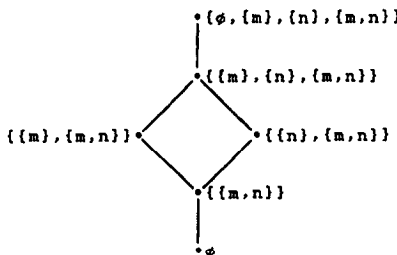


FIG. 2. Distributive lattice for $\Psi(\{m, n\})$.

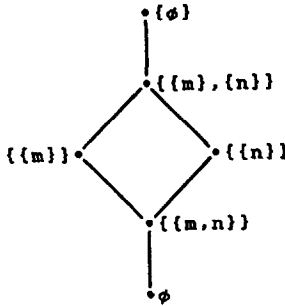


FIG. 3. Distributive lattice for $\Omega(\{m, n\})$.

DEFINITION. Let N be a finite set. $\Omega(N)$ is then the set of $X \subseteq \mathcal{P}(N)$ such that for all $P, Q \in X: P \subseteq Q \Rightarrow P = Q$. For $X, Y \in \Omega(N)$,

$$\begin{aligned}
 X \vee Y &= \min(X \cup Y) \\
 X \wedge Y &= \min(\{P \cup Q \mid P \in X \text{ and } Q \in Y\}) \\
 \bar{X} &= \min(\{Q \subseteq N \mid \forall P \in X: P \cap Q = \emptyset\}).
 \end{aligned}$$

PROPERTY 3. Let N be a finite set. Then \min is an isomorphism from $\langle \Psi(N), \cup, \cap, \bar{} \rangle$ to $\langle \Omega(N), \vee, \wedge, \bar{} \rangle$.

PROPERTY 4. For $X, Y \in \Omega(N)$, $X \leq Y$ if and only if for each P in X , there exists Q in Y such that $Q \subseteq P$.

EXAMPLE. The distributive lattice for $\Omega(\{m, n\})$ is shown in Fig. 3.

4. TOTAL RELATIONS

The Ψ set defined above is used now in establishing a necessary and sufficient condition for a product relation to be total.

PROPERTY 5. Let A, B , and C be sets, let ρ be a relation on $A \times B$, and let σ be a relation on $B \times C$. Then

$$\begin{aligned}
 \{P \subseteq B \mid \rho^{-1}(P) = A\} & \quad \text{and} & \quad \{Q \subseteq B \mid \exists a \in A: \rho(a) \subseteq Q\} \\
 \{P \subseteq B \mid \sigma(P) = C\} & \quad \text{and} & \quad \{Q \subseteq B \mid \exists c \in C: \sigma^{-1}(c) \subseteq Q\}
 \end{aligned}$$

are elements of $\Psi(B)$.

LEMMA 1. *Let A and B be sets and let ρ be a relation on $A \times B$. Then*

$$\overline{\{P \subseteq B \mid \rho^{-1}(P) = A\}} = \{Q \subseteq B \mid \exists a \in A: \rho(a) \subseteq Q\}.$$

Proof. Let

$$X = \{P \subseteq B \mid \rho^{-1}(P) = A\}$$

$$Y = \{Q \subseteq B \mid \exists a \in A: \rho(a) \subseteq Q\}.$$

Suppose that there exists $R \in \bar{X}$ such that

$$\forall a \in A: \exists b \in B: \quad a \in \rho^{-1}(b) \text{ and } b \notin R.$$

Hence $\rho^{-1}(B - R) = A$ and $(B - R) \in X$. But because $R \in \bar{X}$, $R \cap (B - R) \neq \emptyset$ —a contradiction. Thus for all $R \in \bar{X}$,

$$\exists a \in A: \forall b \in B: \quad a \in \rho^{-1}(b) \Rightarrow b \in R.$$

Or equivalently, for all $R \in \bar{X}$,

$$\exists a \in A: \quad \rho(a) \subseteq R.$$

In other words, $\bar{X} \subseteq Y$.

From the definition of Y , it follows that for all $Q \in Y$, there exists $a \in A$ such that $\rho(a) \subseteq Q$. From the definition of X , it follows that for all $P \in X$, for all $a \in A$, $P \cap \rho(a) \neq \emptyset$. Hence for all $P \in X$, for all $Q \in Y$, $P \cap Q \neq \emptyset$. And so $Y \subseteq \bar{X}$.

THEOREM 2. *Let A , B , and C be sets, let ρ be a relation on $A \times B$, and let σ be a relation on $B \times C$. Then $\rho \circ \sigma$ is total if and only if*

$$\overline{\{P \subseteq B \mid \rho^{-1}(P) = A\}} \subseteq \{Q \subseteq B \mid \sigma(Q) = C\}.$$

Proof.

$\rho \circ \sigma$ is total

if and only if

$$\forall a \in A: \quad \sigma(\rho(a)) = C$$

if and only if

$$\forall a \in A: \exists Q \subseteq B: \quad Q \subseteq \rho(a) \text{ and } \sigma(Q) = C$$

if and only if

$$\forall P \subseteq B: \quad (\exists a \in A: P = \rho(a)) \Rightarrow (\exists Q \subseteq B: \sigma(Q) = C \text{ and } Q \subseteq P)$$

if and only if

$$\forall P \subseteq B: \quad (\exists a \in A: \rho(a) \subseteq P) \Rightarrow (\exists Q \subseteq B: \sigma(Q) = C \text{ and } Q \subseteq P)$$

if and only if

$$\{P \subseteq B \mid \exists a \in A: \rho(a) \subseteq P\} \subseteq \{Q \subseteq B \mid \sigma(Q) = C\}$$

if and only if (by Lemma 1)

$$\overline{\{P \subseteq B \mid \rho^{-1}(P) = A\}} \subseteq \{Q \subseteq B \mid \sigma(Q) = C\}.$$

In those cases where B is finite, Property 3 permits the condition in Theorem 2 to be replaced by the corresponding condition involving elements from $\Omega(B)$.

COROLLARY 1. *Let A, B , and C be sets with B finite, let ρ be a relation on $A \times B$, and let σ be a relation on $B \times C$. Then $\rho \circ \sigma$ is total if and only if*

$$\overline{\min(\{P \subseteq B \mid \rho^{-1}(P) = A\})} \leq \min(\{Q \subseteq B \mid \sigma(Q) = C\}).$$

EXAMPLE. Corollary 1 and Property 4 are applied to the two situations in Fig. 1. For the relations in Fig. 1a,

$$\min(\{P \subseteq B \mid \rho^{-1}(P) = A\}) = \{\{m, n\}, \{n, o\}\}$$

$$\overline{\min(\{P \subseteq B \mid \rho^{-1}(P) = A\})} = \{\{n\}, \{m, o\}\}$$

$$\min(\{P \subseteq B \mid \sigma(P) = C\}) = \{\{m, o\}, \{n, o\}\}.$$

In this case,

$$\overline{\min(\{P \subseteq B \mid \rho^{-1}(P) = A\})} \not\leq \min(\{Q \subseteq B \mid \sigma(Q) = C\})$$

and $\rho \circ \sigma$ is not total. For the relations in Fig. 1b,

$$\min(\{P \subseteq B \mid \rho^{-1}(P) = A\}) = \{\{m, n\}, \{n, o\}\}$$

$$\overline{\min(\{P \subseteq B \mid \rho^{-1}(P) = A\})} = \{\{n\}, \{m, o\}\}$$

$$\min(\{P \subseteq B \mid \sigma(P) = C\}) = \{\{n\}, \{m, o\}\}.$$

Here,

$$\overline{\min(\{P \subseteq B \mid \rho^{-1}(P) = A\})} \leq \min(\{Q \subseteq B \mid \sigma(Q) = C\})$$

and $\rho \circ \sigma$ is total.

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